

# ALMOST ALL SETS OF $d + 2$ POINTS ON THE $(d - 1)$ -SPHERE ARE NOT SUBTRANSITIVE

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ABSTRACT. We generalise an argument of Leader, Russell, and Walters to show that almost all sets of  $d + 2$  points on the  $(d - 1)$ -sphere  $S^{d-1}$  are not contained in a transitive set in some  $\mathbf{R}^n$ .

A finite subset of  $\mathbf{R}^d$  is called *transitive* if it has a transitive group of symmetries, and *subtransitive* if it is a subset of a transitive set in some  $\mathbf{R}^n$ , where possibly  $n > d$ . Clearly every subtransitive set lies on a sphere. The converse was answered negatively by Leader, Russell, and Walters [3, 4] in connection to some conjectures in Euclidean Ramsey theory. Their key idea in [3] was to show more strongly that almost all cyclic quadrilaterals are not even *affinely subtransitive*; that is, they do not embed into a transitive set even by a (nonconstant) affine map  $x \mapsto Ax + b$ .

The purpose of this note is to point out that both the result and the argument in [3] generalise straightforwardly: almost all sets of  $d + 2$  points on the  $(d - 1)$ -sphere are not affinely subtransitive. On the other hand, it is not hard to see that every affinely independent set of  $d + 1$  points, in other words a nondegenerate simplex, is subtransitive (see, e.g., [1] or [4]), so this result is best possible.

**Theorem.** Almost every set of  $d + 2$  points on the  $(d - 1)$ -sphere  $S^{d-1} \subset \mathbf{R}^d$  is not affinely subtransitive.

*Proof.* Let  $x_0, \dots, x_{d+1}$  be chosen uniformly at random from  $S^{d-1}$ . Since there are only countably many finite groups, each of which has only countably many orthogonal representations up to orthogonal conjugacy, it suffices to fix a finite subgroup  $G$  of  $O(n)$ , and elements  $g_1, \dots, g_{d+1} \in G$ , and show that almost surely there is no nonconstant affine  $f : \mathbf{R}^d \rightarrow \mathbf{R}^n$  such that

$$f(x_k) = g_k f(x_0) \quad \text{for all } k = 1, \dots, d + 1. \quad (1)$$

If  $x_0, \dots, x_d$  are affinely independent then they may be affinely mapped to the standard affine basis  $0, e_1, \dots, e_d$  of  $\mathbf{R}^d$ . The image  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{R}^d$  of  $x_{d+1}$  is then uniquely determined. The function  $\phi : (x_0, \dots, x_{d+1}) \mapsto \alpha$  thus defined is a rational map, and moreover the image of  $\phi$  is “large”<sup>1</sup>.

<sup>1</sup>For instance,  $\text{image}(\phi) \supset (0, 1)^{d-1} \times (1, \infty)$ . In the language of algebraic geometry,  $\phi$  is a *dominant* rational map: its image is not contained in any proper subvariety.

If  $\phi(x_0, \dots, x_{d+1}) = \alpha$ , then (1) has a nonconstant affine solution if and only if it does when  $(x_0, \dots, x_{d+1})$  is replaced by  $(0, e_1, \dots, e_d, \alpha)$ ; call this condition (1)'. Writing  $f(x) = Ax + b$  with  $A \in \mathbf{R}^{n \times d}$  and  $b \in \mathbf{R}^n$ , the conditions  $k = 1, \dots, d$  of (1)' are equivalent to

$$A = (g_1 b - b, \dots, g_d b - b),$$

while the final condition states that

$$\sum_{k=1}^d \alpha_k (g_k b - b) + b = g_{d+1} b. \quad (2)$$

Note that (2) is an  $n \times n$  linear condition on  $b$ . If the only solutions are fixed points of  $\{g_1, \dots, g_{d+1}\}$ , then  $A$  must be 0, so  $f$  must be constant. Quotienting by the subspace of fixed points of  $\{g_1, \dots, g_{d+1}\}$ , we therefore obtain an  $n \times n'$  (where  $n' \leq n$ ) linear system which has a nonzero solution if and only if (1)' has a nonconstant affine solution. Since each  $n' \times n'$  minor of this system is a polynomial in  $\alpha_1, \dots, \alpha_d$ , it follows, unless each such polynomial is identically zero, that the set of  $\alpha$  such that (1)' has a nonconstant affine solution is contained in a proper subvariety of  $\mathbf{R}^d$ . Since  $\phi$  has large image, it then follows that the set of  $(x_0, \dots, x_{d+1})$  such that (1) has a nonconstant affine solution is contained in a proper subvariety of  $(S^{d-1})^{d+2}$ .

Thus it remains only to show that some  $n' \times n'$  minor of this system is not identically zero, or equivalently that (1)' does not always have a nonconstant affine solution. That is, we must rule out the possibility that  $\{0, e_1, \dots, e_d, \alpha\}$  is affinely subtransitive for every  $\alpha \in \mathbf{R}^d$ . But if  $\alpha = (1/(2d), \dots, 1/(2d))$  then  $\{0, e_1, \dots, e_d, \alpha\}$  is not even convex, so it cannot even be mapped onto a sphere with a nonconstant affine map.  $\square$

For other results about subtransitive sets, see [2].

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## REFERENCES

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